11.1 Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold.

(a) For any smooth function $f : \mathcal{M} \to \mathbb{R}$, we will define the Hessian $Hess[f]$ to be the $(0,2)$-tensor
$$Hess[f] \doteq \nabla df.$$
Show that, in any local coordinate system,
$$Hess[f]_{ij} = \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f.$$
Deduce that $Hess(f)$ is a symmetric tensor. Show also that, for any $p \in \mathcal{M}$ and $X \in T_p\mathcal{M}$,
$$Hess[f](X,X) = \frac{d^2}{dt^2}\big(f \circ \gamma(t)\big)\Big|_{t=0}, \quad \text{where } \gamma \text{ is the geodesic } \gamma(t) = \exp_p(tX).$$

(b) For $f : \mathcal{M} \to \mathbb{R}$, let $c \in \mathbb{R}$ be such that $S = f^{-1}(\{c\})$ is a smooth hypersurface of $\mathcal{M}$ and $df \neq 0$ on S. Show that the scalar second fundamental form $b(\cdot,\cdot)$ of S with respect to the coorientation determined by $\text{grad} f = df^\sharp$ is given by
$$b(X,Y) = -\frac{Hess[f](X,Y)}{\|\text{grad} f\|} \quad \text{for all } X, Y \in \Gamma(\mathcal{M}, S).$$

Solution. (a) Recall that, in any local coordinate system, the covariant derivative of an 1-form ω can be expressed in terms of the Christoffel symbols by the relation
$$(\nabla_i \omega)_j = \partial_i \omega_j - \Gamma^k_{ij} \omega_k.$$
Therefore, for $\omega = df$, we have:
$$Hess[f]_{ij} \doteq (\nabla_i df)_j = \partial_i (df)_j - \Gamma^k_{ij} (df)_k = \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f. \tag{1}$$
The above expression is symmetric in i and j, since $\partial_i \partial_j f = \partial_j \partial_i f$ (f being a smooth function) and $\Gamma^k_{ij} = \Gamma^k_{ji}$ (since the Levi-Civita connection is torsion-free). Therefore, $Hess[f](X,Y) = Hess[f](Y,X)$ for any $X, Y \in \Gamma(\mathcal{M})$.

Let $p \in \mathcal{M}$ and $\gamma(t) = \exp_p(tX)$ (so that $\gamma(0) = p$, $\dot\gamma(0) = X$). Then, in any local coordinate system around p, we calculate using the formula for the derivative of the composition of functions:
$$\begin{aligned}
\frac{d^2}{dt^2}\big(f(\gamma(t))\big) &= \frac{d}{dt}\Big(\partial_i f(\gamma(t)) \cdot \dot\gamma^i(t)\Big) \\
&= \frac{d}{dt}\Big(\partial_i f(\gamma(t))\Big) \cdot \dot\gamma^i(t) + \partial_i f(\gamma(t)) \cdot \ddot\gamma^i(t) \\
&= \partial_i \partial_j f(\gamma(t)) \dot\gamma^j(t) \dot\gamma^i(t) + \partial_i f(\gamma(t)) \cdot \ddot\gamma^i(t).
\end{aligned}$$
Since γ is a geodesic, it satisfies $\ddot\gamma^i(t) = -\Gamma^i_{kl}|_{\gamma(t)} \dot\gamma^k(t) \dot\gamma^l(t)$. Substituting this for the last term in the above relation, we obtain
$$\frac{d^2}{dt^2}\big(f(\gamma(t))\big) = \partial_i \partial_j f(\gamma(t)) \dot\gamma^j(t) \dot\gamma^i(t) - \Gamma^i_{kl}|_{\gamma(t)} \dot\gamma^k(t) \dot\gamma^l(t) \partial_i f(\gamma(t)).$$

Renamiing the summed indices in the second summand (so that $(i,k,l) \to (k,i,j)$), we infer in view of the formula (1):

$$\frac{d^2}{dt^2}\big(f(\gamma(t))\big) = \Big(\partial_i\partial_j f(\gamma(t)) - \Gamma^k_{ij}|_{\gamma(t)}\partial_k f(\gamma(t))\Big)\dot\gamma^i(t)\dot\gamma^j(t) = Hess[f]|_{\gamma(t)}\big(\dot\gamma(t),\dot\gamma(t)\big).$$

Evaluating the above expression at $t=0$, we infer the desired formula.

$$\frac{d^2}{dt^2}\big(f(\gamma(t))\big)\Big|_{t=0} = Hess[f](X,X).$$

(b) For any point $p \in S$ and any $X \in T_p\mathcal{M}$, we have:

$$\langle \mathrm{grad} f, X\rangle = \langle df^\sharp, X\rangle = g_{ij}(df^\sharp)^i X^j = g_{ij}g^{ik}\partial_k f X^j = \delta^k_j \partial_k f X^j = \partial_j f X^j = X(f).$$

Therefore, if X is tangent to $S = f^{-1}(\{c\})$, which is equivalent to the statement that $X(f) = 0$, we must have $\langle \mathrm{grad} f, X\rangle = 0$; in particular, $\mathrm{grad} f|_p \perp T_pS$.

Assuming that $df \neq 0$ on S (and, hence, $\mathrm{grad} f|_S \neq 0$), let us set $\hat n$ to be the unit normal to S in the direction of $\mathrm{grad} f|_S$, namely

$$\hat n \doteq \frac{\mathrm{grad} f}{\|\mathrm{grad} f\|}.$$

For any $p \in S$ and $X, Y \in \Gamma(\mathcal{M}, S)$, we have:

$$b(X,Y) = -\langle \nabla_X \hat n, Y\rangle.$$

Therefore, we can calculate:

$$\begin{aligned} b(X,Y) &= -\Big\langle \nabla_X\big(\frac{\mathrm{grad} f}{\|\mathrm{grad} f\|}\big), Y\Big\rangle \\ &= -\Big\langle \frac{\nabla_X \mathrm{grad} f}{\|\mathrm{grad} f\|} + X\big(\frac{1}{\|\mathrm{grad} f\|}\big)\mathrm{grad} f, Y\Big\rangle \\ &= -\frac{1}{\|\mathrm{grad} f\|}\langle \nabla_X \mathrm{grad} f, Y\rangle - X\big(\frac{1}{\|\mathrm{grad} f\|}\big)\langle \mathrm{grad} f, Y\rangle. \end{aligned}$$

The second term above vanishes (since $\mathrm{grad} f \perp T_pS$); for the first term, we will use the fact that ∇ commutes with the operator $\sharp$ (since $\nabla g = 0$), together with the trivial identity $\langle \omega^\sharp, XY\rangle = \omega(Y)$ to compute

$$\langle \nabla_X \mathrm{grad} f, Y\rangle = \langle \nabla_X(df^\sharp), Y\rangle = \langle (\nabla_X df)^\sharp, Y\rangle = \nabla_X df(Y) = Hess[f](X,Y)$$

(if you do not feel comfortable with manipulating the $\sharp$ operator, you can perform the above calculation directly in local coordinates). Therefore, returning to the previous computation, we infer that

$$b(X,Y) = -\frac{Hess[f](X,Y)}{\|\mathrm{grad} f\|}.$$

11.2 Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold.

(a) The Einstein tensor G of $(\mathcal{M}, g)$ is the $(0,2)$-tensor defined by

$$G = Ric - \frac{1}{2} S g,$$

where S is the scalar curvature of g. Show that G is divergence free, i.e.

$$g^{ab}\nabla_a G_{bc} = 0.$$

(*Hint: You might want to use the second Bianchi identity.*) Deduce that if $(\mathcal{M}, g)$ satisfies

$$Ric = \Lambda g$$

for some smooth function $\Lambda : \mathcal{M} \to \mathbb{R}$ and $\dim\mathcal{M} \geqslant 3$, then $\Lambda = \text{const}$ on each connected component of $\mathcal{M}$ (*Hint: Show first that, in this case, $G = \Lambda' g$ for some different function Λ'*). A Riemannian manifold satisfying such a relation is called an *Einstein manifold.*

(b) Show that if $(\mathcal{M}, g)$ is a connected Einstein manifold of dimension $\dim \mathcal{M} = 3$, then $(\mathcal{M}, g)$ has constant sectional curvature. (*Hint: Exercise 9.1.c might be helpful.*)

Remark. According to the theory of general relativity, a *vacuum* region of our spacetime (i.e. where matter is absent) is modelled by a Lorentzian manifold $(\mathcal{M}, g)$ satisfying $G = \Lambda g$, where Λ is known as the *cosmological constant.* The above results indicate that non-trivial vacuum spacetimes exist only when $\dim\mathcal{M} \geqslant 4$.

Solution. (a) Recall that the Ricci tensor is defined in terms of the Riemann curvature tensor by

$$\text{Ric}_{ij} = g^{ab} R_{aibj}.$$

Note that the above relation is of the form $\text{Ric} = \text{tr}\,\text{tr}(g^{-1} \otimes R)$ (where tr denotes an appropriate contraction). Thus, in view of the facts that $\nabla g = 0$ and ∇ commutes with contractions, we calculate that ∇Ric satisfies

$$g^{ki}\nabla_k \text{Ric}_{ij} = g^{ki} g^{ab} \nabla_k R_{aibj}. \tag{2}$$

Remark. Note that, in the above relation, $\nabla_k \text{Ric}_{ij}$ denotes $(\nabla_{\partial_k \text{Ric}})(\partial_i, \partial_j)$ and similarly $\nabla_k R_{aibj} = (\nabla_{\partial_k} R)(\partial_a, \partial_i, \partial_b, \partial_j)$. It is an easy exercise to verify (using the formula for the coordinate expression of a covariant derivative of a tensor) that $\nabla_k R_{aibj}$ satisfies the same symmetries with respect to the indices (a, i, b, j) as R_{aibj}.

Using the 2^{nd} Bianchi identity

$$\nabla_k R_{aibj} + \nabla_j R_{aikb} + \nabla_b R_{aijk} = 0$$

to substitute the term on the right hand side of (2), we obtain:

$$g^{ki}\nabla_k \text{Ric}_{ij} = -g^{ki} g^{ab} \nabla_j R_{aikb} - g^{ki} g^{ab} \nabla_b R_{aijk}. \tag{3}$$

Using the summetries of the Riemann curvature tensor (see also the remark above) and the fact that ∇ commutes with contractions and with g, we compute the first terms in the right hand side above

$$-g^{ki}g^{ab}\nabla_j R_{aikb} = g^{ki}g^{ab}\nabla_j R_{aibk} = \nabla_j\big(g^{ki}g^{ab}R_{aibk}\big) = \nabla_j S.$$

Similarly, using the symmetries of R we compute that the second term in the right hand side of (3) satisfies

$$-g^{ki}g^{ab}\nabla_b R_{aijk} = -g^{ba}g^{ki}\nabla_b R_{iakj} = g^{ba}\nabla_b \mathrm{Ric}_{aj}$$

where, in the last equality, we used the identity (2) with (b,i,a,k) in place of (k,a,i,b). Therefore, returning to (3), we have

$$g^{ki}\nabla_k \mathrm{Ric}_{ij} = \partial_j S - g^{ba}\nabla_b \mathrm{Ric}_{aj}.$$

Note that (after relabelling the summing indices) the last term in the right hand side above is the same as in the left hand side, so

$$2g^{ki}\nabla_k \mathrm{Ric}_{ij} = \nabla_j S.$$

Using the trivial identity

$$\nabla_j S = \nabla_k(g^{ki}g_{ij}S) = g^{ki}\nabla_k(g_{ij}S),$$

we thus obtain the required relation

$$g^{ki}\nabla_k\big(2\mathrm{Ric}_{ij} - Sg_{ij}\big) = 0. \tag{4}$$

Suppose that $(\mathcal{M}, g)$ satisfies

$$\mathrm{Ric} = \Lambda g.$$

Then, we calculate that

$$S = g^{ab}\mathrm{Ric}_{ab} = \Lambda g^{ab}g_{ab} = n\Lambda,$$

where $n = \dim\mathcal{M}$. Thus, substituting Ric and S in (4), we obtain

$$0 = g^{ki}\nabla_k\big(\mathrm{Ric}_{ij} - \frac{1}{2}Sg_{ij}\big) = g^{ki}\nabla_k\big(\Lambda g_{ij} - \frac{n}{2}\Lambda g_{ij}\big) = \big(1 - \frac{n}{2}\big)g^{ki}g_{ij}\nabla_k\Lambda.$$

Thus, if $n > 2$, we infer that $\partial_j\Lambda = 0$, i.e. Λ is locally constant on $\mathcal{M}$.

(b) In the case when $\dim\mathcal{M} = 3$, we know from Exercise 9.1.c that the full Riemann tensor can be expressed in terms of the Ricci tensor:

$$R_{ijkl} = \mathrm{Ric}_{ik}\, g_{jl} - \mathrm{Ric}_{il}\, g_{jk} + \mathrm{Ric}_{jl}\, g_{ik} - \mathrm{Ric}_{jk}\, g_{il} - \frac{1}{2}S(g_{ik}g_{jl} - g_{jk}g_{il}).$$

Since we assumed that $\mathrm{Ric} = \Lambda g$ for some constant Λ (and, hence, $S = g^{ab}\mathrm{Ric}_{ab} = 3\Lambda$), we deduce after substituting above:

$$\begin{aligned} R_{ijkl} &= \Lambda g_{ik}\, g_{jl} - \Lambda g_{il}\, g_{jk} + \Lambda g_{jl}\, g_{ik} - \Lambda g_{jk}\, g_{il} - \frac{3}{2}\Lambda(g_{ik}g_{jl} - g_{jk}g_{il}) \\ &= \frac{1}{2}\Lambda(g_{ik}g_{jl} - g_{jk}g_{il}). \end{aligned}$$

Therefore, g has constant sectional curvature equal to $\frac{1}{2}\Lambda$.

11.3 Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold.

(a) A 2-dimensional surface $S \subset \mathcal{M}$ is called *ruled* if, for every $p \in \mathcal{M}$, there exists a curve $\gamma : (-\delta, \delta) \to \mathcal{M}$ with $\gamma(0) = p$, $\dot{\gamma}(0) \neq 0$ which is a geodesic of $(\mathcal{M}, g)$ and lies entirely inside S. Show that, in this case,

$$\bar{K}_p \leqslant K[T_pS] \quad \text{for all } \ p \in S,$$

where $\bar{K}_p$ is the sectional curvature of S with respect to the induced metric $\bar{g}$, while $K[T_pS]$ is the sectional curvature of the plane $T_pS \subset T_p\mathcal{M}$ with respect to the ambient metric g. This is known as *Synge's inequality.*

(b) Let q be a point in $\mathcal{M}$ and let $\Omega \subset T_q\mathcal{M}$ be a convex open neighborhood of 0 such that $\exp_q$ is a diffeomorphism when restricted on Ω. Let $S \subset \mathcal{M}$ be the surface defined by $S = \exp_q(\Omega \cap V)$, where V is a 2-dimensional subspace of $T_q\mathcal{M}$. Show that S is a ruled surface. Moreover, show that at the point q:

$$\bar{K}_q = K[T_qS].$$

Solution. (a) Let $B(\cdot, \cdot)$ be the second fundamental form of $S \subset \mathcal{M}$. As we have shown in class, if $\gamma : [a, b] \to S$ is a geodesic in S (i.e. satisfies $\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0$), its acceleration in $\mathcal{M}$ satisfies

$$\nabla_{\dot{\gamma}}\dot{\gamma} = B(\dot{\gamma}, \dot{\gamma}).$$

If S is a *ruled* surface, then, for every $p \in S$, there exists a geodesic γ of $\mathcal{M}$ passing through p which lies entirely in S (hence it is also a geodesic of S, since $\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = \pi^{\top}(\nabla_{\dot{\gamma}}\dot{\gamma}) = 0$; thus, there exists a $v \in T_pS$ (with $v = \dot{\gamma}$) such that

$$B(v, v) = 0.$$

Let us choose for every point $p \in S$ an orthonormal frame $\{e_1, e_2\}$ for T_pS such that $e_1 = v$. Using the Gauss equation, we showed in class that, for the 2-plane $\Pi = T_pS \subset T_p\mathcal{M}$, the sectional curvatures of $(\mathcal{M}, g)$ and $(S, \bar{g})$ at p are related by

$$\bar{K}_p = K_p(\Pi) + \frac{\langle B(e_1, e_1), B(e_2, e_2)\rangle - \|B(e_1, e_2)\|^2}{\|e_1\|^2\|e_2\|^2 - \langle e_1, e_2\rangle^2}.$$

Since $\{e_1, e_2\}$ is orthonormal and $B(e_1, e_1) = 0$, we infer that

$$\bar{K}_p = K_p(\Pi) - \|B(e_1, e_2)\|^2 \leqslant K_p(\Pi).$$

(b) Let $S = \exp_q(V \cap \Omega)$. Since $\exp_q : \Omega \subset T_q\mathcal{M} \to \mathcal{U} = \exp_q(\Omega) \subset \mathcal{M}$ is a diffeomorphism, S is a smooth 2-dimensional surface in $\mathcal{M}$; the map $\Phi = \exp_q|_{V \cap \Omega} : V \cap \Omega \to S$ then defines a smooth parametrization of S (and Φ^{-1} defines a coordinate chart, once we identify V with $\mathbb{R}^2$). Every $p \in S$ is of the form $p = \exp_q(v)$ for some $v \in V \cap \Omega$; in this case, the curve $t[0, 1] \to \gamma(t) = \exp_q(tv)$ is a geodesic of $\mathcal{M}$ that lies entirely in S and passes through p (since Ω was assumed to be a convex neighborhood of 0, $t \cdot v \in \Omega \cap V$ for $t \in [0, 1]$ and, thus, $\exp_q(tv) \in \exp_q(V \cap \Omega) = S$ for all $t \in [0, 1]$).

Therefore, S is a ruled surface. Moreover, at the point q, every direction $v \in T_qS = V$ is a direction through which passes a geodesic of $\mathcal{M}$ lying entirely in S. To see this, note that, since Ω is an open neighborhood of $0 \in T_q\mathcal{M}$, for any $v \in V$ there exists a $\lambda_v > 0$ such that $\lambda_v v \in \Omega \cap V$; in this case, the curve $t \in [0, \lambda_v] \to \gamma(t) = \exp_q(tv)$ is a geodesic of $\mathcal{M}$ that lies entirely in $S = \exp_q(V \cap \Omega)$ and satisfies $\gamma(0) = q$, $\dot{\gamma}(0) = v$. Therefore, as we explained in the previous part of this exersice, the second fundamental form $B(\cdot, \cdot)$ of S satisfies at q:

$$B(v, v) = 0 \quad \text{for all } v \in T_qS = V.$$

Since $B(\cdot, \cdot)$ is symmetric and bilinear, this implies that

$$B(v, w) = 0 \quad \text{for all } v, w \in T_qS = V.$$

Thus, if $\{e_1, e_2\}$ is a basis for $T_qS = V$, we have

$$\bar{K}_q = K_q(T_qS) + \frac{\langle B(e_1, e_1), B(e_2, e_2)\rangle - \|B(e_1, e_2)\|^2}{\|e_1\|^2\|e_2\|^2 - \langle e_1, e_2\rangle^2} = K_q(T_qS).$$

11.4 (a) Let $S \subset (R^3, g_E)$ be a smooth surface which is contained inside the ball

$$B_R = \big\{x \in \mathbb{R}^3 : \|x\| \leqslant R\big\}$$

and such that there exists a point $z \in S$ with $z \in \partial B_R$ (i.e. $\|z\| = R$). Deduce that S and $S_R = \partial B_R$ have the same tangent plane at z. Show that the sectional curvature K of S satisfies at the point z

$$K_z \geqslant \frac{1}{R^2}.$$

Hint: It might be useful to compare the sectional curvatures of S and S_R at z by expressing both surfaces locally as graphs of functions defined over their common tangent plane T_zS and use Exercise 9.1.

(*b) A surface $S \subset \mathbb{R}^3$ is called *minimal* if it has vanishing mean curvature H (such a surface is a stationary point of the total surface functional $\mathcal{A}[S] = \int_S d\bar{g}$, hence the name). Show that a minimal surface satisfies $K \leqslant 0$. Deduce that there is no compact minimal surface in $\mathbb{R}^3$. (*Hint: For a compact minimal surface S, start from a sphere completely surrounding S and decrease its radius until you end up with a sphere both containing S and touching S at a point z.*)

Solution. (a) Let us consider the polar coordinate system (r, θ, ϕ) on $\mathbb{R}^3$ and let us assume, without loss of generality, that the point z does not lie in a region where (r, θ, ϕ) does not degenerate (i.e. at $\theta = 0, \pi$); we can always achieve that by rotating, if necessary, the coordinate system. In this case, the tangent plane T_zS_R of the sphere $S_R = \{r = R\}$ at the point z is spanned by the coordinate vector fields $\partial_\theta|_z, \partial_\phi|_z$. In order to show that $T_zS = T_zS_R$, it suffices to show that, for any curve $t \to \gamma(t)$ inside S with $\gamma(0) = z$, $\dot{\gamma}(0) \in \text{span}\{\partial_\theta|_z, \partial_\phi|_z\}$, i.e. that

$$\dot{\gamma}^r(0) = 0 \quad \Leftrightarrow \quad \frac{d}{dt} r(\gamma(t))\big|_{t=0} = 0.$$

Since $S \subset \{r \leqslant R\}$ and $z \in \{r = R\}$, we infer that, for any curve γ in S as above, we have $r(\gamma(t)) \leqslant R$ for all t and $r(\gamma(0)) = R$; therefore, $\frac{d}{dt} r(\gamma(t))|_{t=0} = 0$. Thus, we have shown that $T_z S = T_z S_R$.

Let us consider a Cartesian coordinate system (x^1, x^2, x^3) on $\mathbb{R}^3$ such that z lies at the center $(0,0,0)$, the 2-plane $T_z S = T_z S_R$ is the coordinate plane $\{x^3 = 0\}$ (hence spanned by $\partial_1|_z, \partial_2|_z$) and the vector $\partial_3|_z$ points in the direction of ∂_r. Since the surfaces S and S_R are smooth and ∂_3 is transversal to $T_z S$, $T_z S_R$ at z, it will also be transversal to to $T_p S$, $T_q S_R$ for $p \in S$ and $q \in S_R$ close enough to z. Therefore, we can express both surfaces as graphs of functions over the (x^1, x^2) coordinate plane in a small neighborhood around z, i.e. there exist smooth functions $F, F_R : B_\delta(0) \subset \mathbb{R}^2 \to \mathbb{R}$ (for some $\delta > 0$ small enough) and open neighborhoods $\mathcal{U} \subset S$, $\mathcal{V} \subset S_R$ of z, such that:

$$S \cap \mathcal{U} = \big\{x^3 = F(x^1, x^2),\, (x^1, x^2) \in B_\delta(0)\big\}, \quad S_R \cap \mathcal{V} = \big\{x^3 = F_R(x^1, x^2),\, (x^1, x^2) \in B_\delta(0)\big\}.$$

Note that since $z = (0,0,0)$ belongs to both surfaces, we have

$$F(0,0) = 0 = F_R(0,0).$$

Moreover, since the plane $\{x^3 = 0\}$ is tangent to S, S_R at z, we also have

$$\partial_i F(0,0) = 0 = \partial_i F_R(0,0), \quad i = 1,2.$$

Finally, note that, since S lies in the interior of the ball S_R and the coordinate vector $\partial_3|_z$ points in the direction of ∂_r, the functions F and F_R satisfy:

$$F(x^1, x^2) \leqslant F_R(x_1, x_2) \leqslant 0$$

(note that $F_R(x_1, x_2) \leqslant 0$ because the ball S_R lies on one side of the hyperplane $T_z S_R = \{x^3 = 0\}$, namely in the half space $\{x^3 \leqslant 0\}$). The above conditions imply that the 2×2 symmetric matrices $[\partial_i \partial_j F](0,0)$ and $[\partial_i \partial_j F_R](0,0)$ satisfy

$$[\partial_i \partial_j F](0,0) \leqslant [\partial_i \partial_j F]_R(0,0) \leqslant 0 \tag{5}$$

(recall that two symmetric $n \times n$ matrices A, B satisfy $A \leqslant B$ if $x^T A x \leqslant x^T B x$ for any vector x). Note also that

Using the last relation established in the solution of Exercise 9.1, we can compute the Riemann curvature tensors of S and S_R (equipped with the corresponding induced metrics $\bar{g}$ and $\bar{g}_R$) in the (x^1, x^2) coordinate systems by the formulas

$$(R_{\bar{g}})_{ijkl} \frac{\partial_i \partial_k F \cdot \partial_j \partial_l F - \partial_i \partial_l F \cdot \partial_j \partial_k F}{1 + |dF|^2}$$

and

$$(R_{\bar{g}_R})_{ijkl} \frac{\partial_i \partial_k F_R \cdot \partial_j \partial_l F_R - \partial_i \partial_l F_R \cdot \partial_j \partial_k F_R}{1 + |dF_R|^2}$$

Therefore, using the formula defining the sectional curvature, we can evaluate at $(x^1, x^2) = (0,0)$ (where $dF = dF_R = 0$ and $\bar{g}_{ij}|_{(0,0)} = (\bar{g}_R)_{ij}|_{(0,0)} = \delta_{ij}$:

$$K_{\bar{g}}|_z = \det\big([\partial_i \partial_j F](0,0)\big), \quad K_{\bar{g}_R}|_z = \det\big([\partial_i \partial_j F_R](0,0)\big).$$

In view of the relation (5) between the matrices $[\partial_i\partial_j F](0,0)$ and $[\partial_i\partial_j F_R](0,0)$, we infer that

$$K_{\bar{g}}|_z \geqslant K_{\bar{g}_R}|_z.$$

Since $(S_R, \bar{g}_R)$ is the round sphere of radius R, we have $K_{\bar{g}_R}|_z = \frac{1}{R^2}$, hence

$$K_{\bar{g}}|_z \geqslant \frac{1}{R^2}.$$

(b) Let p be any point on $S \subset \mathbb{R}^3$ and $\{e_1, e_2\}$ be an orthonormal base of T_pS. Let b be the scalar second fundamental form of S (with respect to a fixed unit normal $\hat{n}$ to S) and let us define the symmetric 2×2 matrix

$$B = \begin{pmatrix} b(e_1, e_1) & b(e_1, e_2) \\ b(e_2, e_1) & b(e_2, e_2) \end{pmatrix}.$$

The sectional curvature $K|_p$ of S equipped with the induced metric satisfies (in view of the Gauss equation and the fact that the Riemann curvature tensor of $(\mathbb{R}^3, g_E)$ vanishes identically)

$$K|_p = b(e_1, e_1)b(e_2, e_2) - \big(b(e_1, e_2)\big)^2 = \det B,$$

while the mean curvature $H|_p$ was defined so that

$$H|_p = b(e_1, e_1) + b(e_2, e_2) = \mathrm{tr} B.$$

Notice that B is diagonalizable with real eigenvalues (since it is symmetric), so if $\lambda_1, \lambda_2 \in \mathbb{R}$ are its eigenvalues, we have $\mathrm{tr}B = \lambda_1 + \lambda_2$, $\det B = \lambda_1\lambda_2$. From this it readily follows that if $\mathrm{tr}B = 0$, then $\det B \leqslant 0$. In particular, if S is a minimal surface, then $K|_p \leqslant 0$ for all $p \in \mathcal{S}$.

We will show that there is no compact minimal surface in $\mathbb{R}^3$ by contradiction: Assume that such a minimal surface S existed. Since S is compact it is also bounded. Let

$$R = \min\{\rho > 0 : \ S \subset B_\rho(0)\},$$

that is to say, $B_R(0)$ is a closed ball of minmal radius which contains S entirely. Let us set $S_R = \partial B_R(0) = \{r = R\}$. Note that there exists a point $z \in S_R$ such that $z \in S$: If this is not the case, i.e. if $S \cap S_R = \emptyset$, then, due to the compactness of S, we must have $\max_S r < R$; in this case, there would exist a $\delta > 0$ such that $S \subset \{r \leqslant R - \delta\}$, thus violating the assumption that R is the minimal value with this property. From part (a) of this exercise, it would then follow that

$$K|_z \geqslant \frac{1}{R^2},$$

which contradicts the fact that $K \leqslant 0$ everywhere on a minimal surface.

* **11.5** Let $\gamma : [0,1] \to \mathcal{M}$ be a geodesic of $(\mathcal{M}, g)$. Assume that there exist points $0 < a < b < 1$ and a vector field Z along γ with $Z \perp \dot{\gamma}$ satisfying the Jacobi equation

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}Z - R(\dot{\gamma}, Z)\dot{\gamma} = 0$$

and such that

$$Z(a) = Z(b) = 0$$

with Z not identically 0 on $[a, b]$. Show that γ cannot be length minimizing among all curves connecting $\gamma(0)$ to $\gamma(1)$. *(Hint: You have to construct a variation ϕ_s of $\phi_0 = \gamma$ fixing the endpoints of γ such that $\frac{d^2}{ds^2}(\ell(\phi_s))|_{s=0} < 0$. To this end, consider first the variation determined by a variation vector field which is equal to Z in $[a, b]$ and 0 otherwise, and then consider small perturbations of this vector field around $t = a, b$.)*

Solution. First of all, since Z is not identically 0 along γ, we must have $\nabla_{\dot{\gamma}} Z(a) \neq 0$ and $\nabla_{\dot{\gamma}} Z(b) \neq 0$ (this can be seen via a contradiction argument: If $\nabla_{\dot{\gamma}} Z(a) = 0$, then Z would satisfy the same initial coditions at $t = a$ as the zero vector field; since the Jacobi equation is a linear second order ODE, the uniqueness property of solutions to the corresponding initial value problem at $t = 1$ would imply that $Z \equiv 0$, which is a contradiction; similarly at $t = b$). Therefore, for $\delta > 0$ sufficiently small (to be determined more precisely later), we have $Z(a + \delta) \neq 0$ and $Z(b - \delta) \neq 0$. Let us define the following auxiliary unit vector field E on $\gamma|_{[a-\delta,a+\delta]\cup[b-\delta,b+\delta]}$:

1. For $t \in [a - \delta, a + \delta]$, we will define $E(t)$ by parallel transporting $\frac{Z(a+\delta)}{\|Z(a+\delta)\|}$, i.e.

$$\begin{cases} \nabla_{\dot{\gamma}} E = 0, \\ E(a + \delta) = \frac{Z(a+\delta)}{\|Z(a+\delta)\|}. \end{cases}$$

2. For $t \in [b - \delta, b + \delta]$, we will define $E(t)$ similarly by

$$\begin{cases} \nabla_{\dot{\gamma}} E = 0, \\ E(b - \delta) = \frac{Z(b+\delta)}{\|Z(b+\delta)\|}. \end{cases}$$

Since E and $\dot{\gamma}$ are both parallel transported on the $t \in [a-\delta, a+\delta]$ and satisfy $E(a+\delta) \perp \dot{\gamma}(a+\delta)$ (due to our assumption on Z), we must have $E(t) \perp \dot{\gamma}(t)$ for all $t \in [a - \delta, a + \delta]$. Similarly, $E(t) \perp \dot{\gamma}(t)$ for all $t \in [b - \delta, b + \delta]$ (since $E(b - \delta) \perp \dot{\gamma}(b - \delta)$).

Finally, let us define the function $f : [a - \delta, a + \delta] \cup [b - \delta, b + \delta]$ by the relation

$$f(t) = \begin{cases} \frac{t-a+\delta}{2\delta}\|Z(a + \delta)\|, & t \in [a - \delta, a + \delta], \\ \frac{b+\delta-t}{2\delta}\|Z(b - \delta)\|, & t \in [b - \delta, b + \delta] \end{cases}$$

(note that $f(a - \delta) = f(b + \delta) = 0$, $f(a + \delta) = \|Z(a + \delta)\|$, $f(b - \delta) = \|Z(b - \delta)\|$). Using the above ingredients, we will define the following vector field along γ which can be thought of as a "perturbation" of Z on $\gamma|_{[a,b]}$:

$$\tilde{Z}(t) = \begin{cases} Z(t), & t \in (a + \delta, b - \delta), \\ f(t)E(t), & t \in [a - \delta, a + \delta] \cup [b - \delta, b + \delta], \\ 0, & t \in [0, a - \delta) \cup (b + \delta, 1]. \end{cases}$$

Note that $\tilde{Z}$ is continuous and piecewise smooth for $t \in [0,1]$; $\nabla_{\dot{\gamma}}\tilde{Z}$ has a jump discontinuity at $t = a \pm \delta, b \pm \delta$. Moreover, $\tilde{Z} \perp \dot{\gamma}$.

Let us consider the variation of $\gamma(t)$ defined by

$$\phi(s,t) = \exp_{\gamma(t)}\big(s\tilde{Z}(t)\big).$$

Notice that $\phi(s,t)$ satisfies the following properties:

1. $\phi(0,t) = \gamma(t)$ and $\phi(s,0) = \gamma(0)$, $\phi(s,1) = \gamma(1)$ for all s (since $\tilde{Z}(0) = 0$ and $\tilde{Z}(1) = 0$.
2. For any $t \in [0,1]$, the curves $s \to \phi(s,t)$ are geodesics; hence, the variation vector field

$$X = \frac{\partial \phi}{\partial s}$$

satisfies $\nabla_X X = 0$.

Therefore, applying the second variation formula for the length of the curves $t \to \phi_s(t) = \phi(s,t)$, we obtain

$$\begin{aligned}
\frac{d^2}{ds^2}\ell(\phi_s)\Big|_{s=0} &= \langle \nabla_X X, \dot{\gamma}\rangle\big|_{t=0}^{t=1} + \int_0^1 \Big(\|\nabla_{\dot{\gamma}}\tilde{Z}^{\perp}\|^2 - R(\dot{\gamma},\tilde{Z},\dot{\gamma},\tilde{Z})\Big)\,dt \\
&= 0 + \int_0^1 \Big(\|\nabla_{\dot{\gamma}}\tilde{Z}\|^2 - R(\dot{\gamma},\tilde{Z},\dot{\gamma},\tilde{Z})\Big)\,dt \\
&= \int_{a-\delta}^{a+\delta} \Big(\|\nabla_{\dot{\gamma}}\tilde{Z}\|^2 - R(\dot{\gamma},\tilde{Z},\dot{\gamma},\tilde{Z})\Big)\,dt + \int_{a+\delta}^{b-\delta} \Big(\|\nabla_{\dot{\gamma}}\tilde{Z}\|^2 - R(\dot{\gamma},\tilde{Z},\dot{\gamma},\tilde{Z})\Big)\,dt \\
&\quad + \int_{b-\delta}^{b+\delta} \Big(\|\nabla_{\dot{\gamma}}\tilde{Z}\|^2 - R(\dot{\gamma},\tilde{Z},\dot{\gamma},\tilde{Z})\Big)\,dt.
\end{aligned}$$

Remark. Even though we established the second variation formula in class for smooth variation vector fields, it is also valid in the piecewise smooth setting. One way to see that (apart from going through the details of the proof) is by applying the formula for a sequence of smooth approximations of a given continuous and piecewise smooth variation vector field.

We will now compute the three integrals appearing in the right hand side above separately:

1. In the interval $t \in [a-\delta, a+\delta]$, we have $\tilde{Z}(t) = f(t)E(t)$. Since $\nabla_{\dot{\gamma}}E = 0$, we have $\nabla_{\dot{\gamma}}\tilde{Z}(t) = f'(t)E(t)$. Therefore, using the expression for f on $[a-\delta, a+\delta]$, we calculate

$$\begin{aligned}
&\int_{a-\delta}^{a+\delta} \Big(\|\nabla_{\dot{\gamma}}\tilde{Z}\|^2 - R(\dot{\gamma},\tilde{Z},\dot{\gamma},\tilde{Z})\Big)\,dt \\
&\qquad = \int_{a-\delta}^{a+\delta} \Big((f'(t))^2\|E(t)\|^2 - (f(t))^2 R(\dot{\gamma},E,\dot{\gamma},E)\Big)\,dt \\
&\qquad \overset{\|E\|=1}{=} \int_{a-\delta}^{a+\delta} \Big(\Big[\frac{1}{2\delta}\|Z(a+\delta)\|\Big]^2 - \Big[\frac{t-a+\delta}{2\delta}\|Z(a+\delta)\|\Big]^2 R(\dot{\gamma},E,\dot{\gamma},E)\Big)\,dt
\end{aligned}$$

$$= \Big(\int_{a-\delta}^{a+\delta} \frac{1}{4\delta^2}\, dt - \int_{a-\delta}^{a+\delta} \Big[\frac{t-a+\delta}{2\delta}\Big]^2 K(\dot\gamma, E)\|\dot\gamma\|^2\Big)\, dt\Big)\|Z(a+\delta)\|^2$$
$$= \Big(\frac{1}{2\delta} - \int_{a-\delta}^{a+\delta} \Big[\frac{t-a+\delta}{2\delta}\Big]^2 K(\dot\gamma, E)\|\dot\gamma\|^2\Big)\, dt\Big)\|Z(a+\delta)\|^2.$$

Notice that, if $\mathcal{U}$ is an open neighborhood of $\gamma(a)$ in $\mathcal{M}$ with compact closure then, provided δ is small enough so that $\gamma|_{[a-\delta,a+\delta]} \subset \mathcal{U}$, we can bound

$$\Big| \int_{a-\delta}^{a+\delta} \Big[\frac{t-a+\delta}{2\delta}\Big]^2 K\dot\gamma, E)\|\dot\gamma\|^2\Big)\, dt\Big| \leqslant \max_{p\in\bar{\mathcal{U}}, \Pi\subset T_p\mathcal{M}} |K(\Pi)| \int_{a-\delta}^{a+\delta} \Big[\frac{t-a+\delta}{2\delta}\Big]^2 dt$$
$$= \max_{p\in\bar{\mathcal{U}}, \Pi\subset T_p\mathcal{M}} |K(\Pi)| \cdot \frac{2\delta}{3}.$$

Therefore, as $\delta \to 0$,

$$\int_{a-\delta}^{a+\delta} \Big(\|\nabla_{\dot\gamma}\tilde Z\|^2 - R(\dot\gamma, \tilde Z, \dot\gamma, \tilde Z)\Big)\, dt = \Big(\frac{1}{2\delta} + O(\delta)\Big)\|Z(a+\delta)\|^2.$$

2. Arguing in exactly the same way for $t \in [b-\delta, b+\delta]$, we obtain

$$\int_{b-\delta}^{b+\delta} \Big(\|\nabla_{\dot\gamma}\tilde Z\|^2 - R(\dot\gamma, \tilde Z, \dot\gamma, \tilde Z)\Big)\, dt = \Big(\frac{1}{2\delta} + O(\delta)\Big)\|Z(b-\delta)\|^2.$$

3. In the interval $t \in [a+\delta, b-\delta]$, we have $\tilde Z = Z$. Therefore,

$$\begin{aligned}\int_{a+\delta}^{b-\delta} \Big(\|\nabla_{\dot\gamma}\tilde Z\|^2 - R(\dot\gamma, \tilde Z, \dot\gamma, \tilde Z)\Big)\, dt &= \int_{a+\delta}^{b-\delta} \Big(\langle\nabla_{\dot\gamma} Z, \nabla_{\dot\gamma} Z\rangle - R(\dot\gamma, \tilde Z, \dot\gamma, \tilde Z)\Big)\, dt\\ &= \int_{a+\delta}^{b-\delta} \Big(\frac{d}{dt}\langle Z, \nabla_{\dot\gamma} Z\rangle - \langle Z, \nabla_{\dot\gamma}\nabla_{\dot\gamma} Z\rangle - R(\dot\gamma, \tilde Z, \dot\gamma, \tilde Z)\Big)\, dt\\ &= \langle Z, \nabla_{\dot\gamma} Z\rangle|_{t=a+\delta}^{b-\delta} - \int_{a+\delta}^{b-\delta} \Big(\langle Z, \nabla_{\dot\gamma}\nabla_{\dot\gamma} Z\rangle - R(\dot\gamma, \tilde Z, \dot\gamma, \tilde Z)\Big)\, dt\\ &= \langle Z, \nabla_{\dot\gamma} Z\rangle|_{t=a+\delta}^{b-\delta} - \int_{a+\delta}^{b-\delta} \Big(\langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} Z, Z\rangle + \langle R(\dot\gamma, \tilde Z)\dot\gamma, \tilde Z\rangle\Big)\, dt.\end{aligned}$$

Using the fact that Z solves the Jacobi equation $\nabla_{\dot\gamma}\nabla_{\dot\gamma} Z - R(\dot\gamma, Z)\dot\gamma = 0$, we therefore deduce that

$$\int_{a+\delta}^{b-\delta} \Big(\|\nabla_{\dot\gamma}\tilde Z\|^2 - R(\dot\gamma, \tilde Z, \dot\gamma, \tilde Z)\Big)\, dt = \langle Z, \nabla_{\dot\gamma} Z\rangle|_{t=a+\delta}^{t=b-\delta}.$$

Returning to the expression for $\frac{d^2}{ds^2}\ell(\phi_s)\Big|_{s=0}$ and substituting the above relations for the three integrals in the right hand side, we obtain:

$$\frac{d^2}{ds^2}\ell(\phi_s)\Big|_{s=0} = \Big(\frac{1}{2\delta} + O(\delta)\Big)\|Z(a+\delta)\|^2 + \langle Z, \nabla_{\dot\gamma} Z\rangle|_{t=a+\delta}^{t=b-\delta} + \Big(\frac{1}{2\delta} + O(\delta)\Big)\|Z(b-\delta)\|^2$$

$$
\begin{aligned}
&= \frac{1}{2\delta}\|Z(a+\delta)\|^2 - \langle Z(a+\delta), \nabla_{\dot\gamma} Z(a+\delta)\rangle \\
&\quad + \frac{1}{2\delta}\|Z(b-\delta)\|^2 + \langle Z(b-\delta), \nabla_{\dot\gamma} Z(b-\delta)\rangle + O(\delta)\big(\|Z(a+\delta)\|^2 + \|Z(b-\delta)\|^2\big) \\
&= \frac{1}{2\delta}\Big\langle Z(a+\delta), Z(a+\delta) - 2\delta\nabla_{\dot\gamma} Z(a+\delta)\Big\rangle \\
&\quad + \frac{1}{2\delta}\Big\langle Z(b-\delta), Z(b-\delta) + 2\delta\nabla_{\dot\gamma} Z(b-\delta)\Big\rangle + O(\delta)\big(\|Z(a+\delta)\|^2 + \|Z(b-\delta)\|^2\big).
\end{aligned}
$$

Using Taylor's theorem (and the fact that $Z(a) = 0$), we can express as $\delta \to 0$:

$$
Z(a+\delta) = Z(a) + \delta\nabla_{\dot\gamma} Z(a) + O(\delta^2) = \delta\nabla_{\dot\gamma} Z(a) + O(\delta^2)
$$

and

$$
\nabla_{\dot\gamma} Z(a+\delta) = \nabla_{\dot\gamma} Z(a) + O(\delta).
$$

Therefore,

$$
\begin{aligned}
\Big\langle Z(a+\delta), Z(a+\delta) - 2\delta\nabla_{\dot\gamma} Z(a+\delta)\Big\rangle &= \Big\langle \delta\nabla_{\dot\gamma} Z(a) + O(\delta^2), \delta\nabla_{\dot\gamma} Z(a) + O(\delta^2) - 2\delta\nabla_{\dot\gamma} Z(a) + O(\delta^2)\Big\rangle \\
&= -\delta^2\langle \nabla_{\dot\gamma} Z(a), \nabla_{\dot\gamma} Z(a)\rangle + O(\delta^3).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
Z(b-\delta) &= -\delta\nabla_{\dot\gamma} Z(b) + O(\delta^2), \\
\nabla_{\dot\gamma} Z(b-\delta) &= \nabla_{\dot\gamma} Z(b) + O(\delta)
\end{aligned}
$$

and

$$
\Big\langle Z(b-\delta), Z(b-\delta) + 2\delta\nabla_{\dot\gamma} Z(b-\delta)\Big\rangle = -\delta^2\langle \nabla_{\dot\gamma} Z(b), \nabla_{\dot\gamma} Z(b)\rangle + O(\delta^3).
$$

Therefore,

$$
\frac{d^2}{ds^2}\ell(\phi_s)\Big|_{s=0} = -\frac{\delta}{2}\Big(\|\nabla_{\dot\gamma} Z(a)\|^2 + \|\nabla_{\dot\gamma} Z(b)\|^2\Big) + O(\delta^2).
$$

Since $\nabla_{\dot\gamma} Z(a), \nabla_{\dot\gamma} Z(b) \neq 0$, we infer that, choosing $\delta > 0$ sufficiently small, we have

$$
\frac{d^2}{ds^2}\ell(\phi_s)\Big|_{s=0} < 0.
$$

Since $\frac{d}{ds}\ell(\phi_s)\Big|_{s=0} = 0$ (because γ is a geodesic), this implies that $\ell(\phi_s) < \ell(\gamma)$ for $s \neq 0$ small enough.